Ground state in the limit $h/J \gg 1$:

This limit is much more intersting. First set $J = 0$.

$$H = - h \sum_{\square} \prod_{\square} Z \quad \text{with} \quad \prod_{+} X = 1.$$

The Gauss law constraint commutes with H

$$\Rightarrow \quad H \equiv - h \sum_{\square} \prod_{\square} Z - \sum_{+} \prod_{+} X$$

i.e. the constraint can be implemented energetically if one is interested only in the ground state and low lying excitations. One can even set $h = 1$. Therefore, we consider

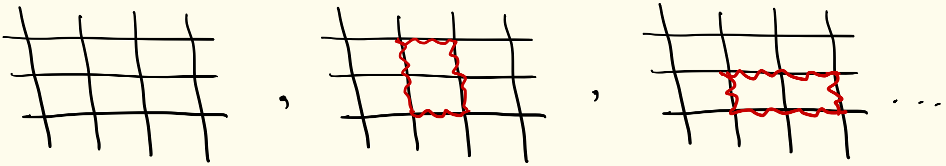
$$H = - \underbrace{\sum_{\square} \prod_{\square} Z}_{H_P} - \underbrace{\sum_{+} \prod_{+} X}_{H_S}.$$

This is Kitaev's toric code model.

Ground state on a plane:

To find the ground state, lets first work in the X-basis. Any configuration that minimizes the H_S term corresponds to a closed loop config.

e.g.



where red line indicates $X=-1$ on the corresponding link. Otherwise $X=+1$. Acting with a single term in H_P e.g. $\prod_{\square} Z_i$ on a specific site generates another closed loop config (this is because H_S and H_P commute).

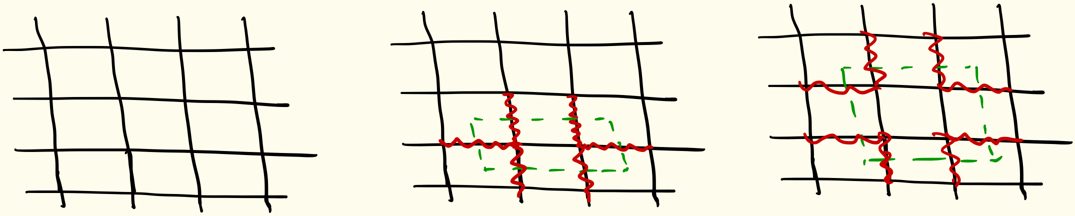
It is natural to guess that a ground state is a superposition of such close loop configs. Infact, the only way to make it work is to superpose closed loop configs. with equal weight:

$$|g.s.\rangle = \sum_{\mathcal{D}} |\mathcal{D}\rangle$$

One way to write this explicitly is: Chose a reference state $|\psi_0\rangle = \prod_i |\rightarrow\rangle_i$ ie. all spins pointing along $+X$-direction. Then,

$$\boxed{|g.s.\rangle = \prod_{\square}\left[1 + \prod_{\square} Z_i\right]|\psi_0\rangle}$$

One can also work in the Z-basis. The configs. that minimize H_p look like:



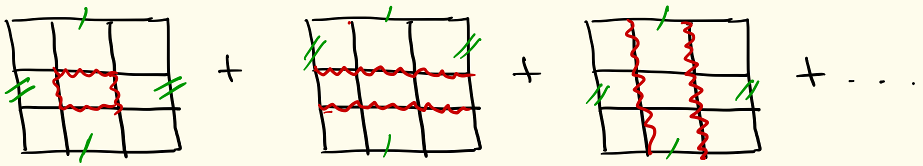
where red links correspond to $Z=-1$. Again, they are closed loop configs. but now on the dual lattice. The g.s. in this basis can be written as:

$$|g.s.\rangle = \prod_{+}\left[1 + \prod_{+} X\right]|\psi_0'\rangle \quad \text{where}$$

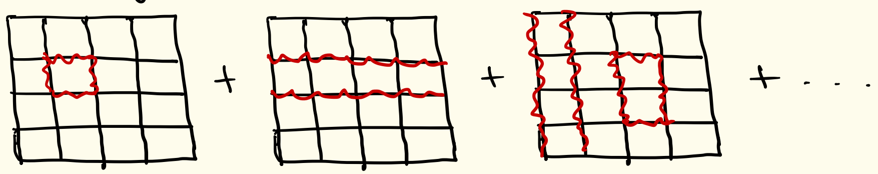
$|\psi_0'\rangle = \prod_i |\uparrow\rangle_i$ i.e. all-up config.

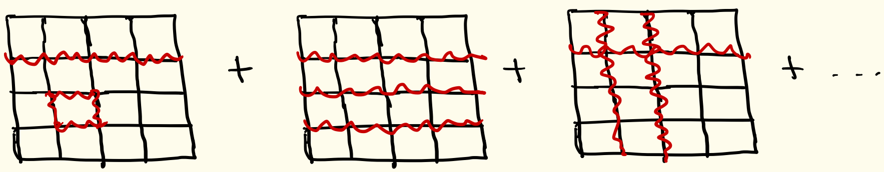
Ground states on a torus:

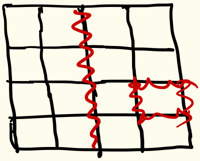
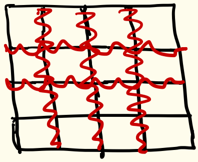
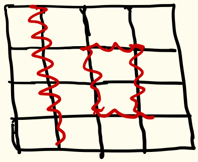
One important property of the wave-fn of the form $\prod_{\square}[1+\prod_{\square} Z]|\psi_0\rangle$ or $\prod_{+}[1+\prod_{+} X]|\psi_0'\rangle$ is that it only superposes closed loop configs. that have identical parity of the winding number along non-contractible cycles on a torus:

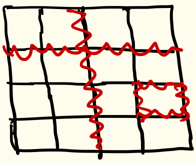
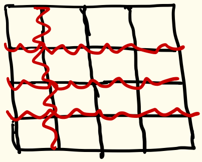
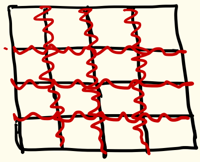


Therefore one can construct $\underline{\underline{4}}$ orthogonal ground states on a torus, corresponding to different parity of the winding along x, y directions

$|\psi_{00}\rangle =$ 

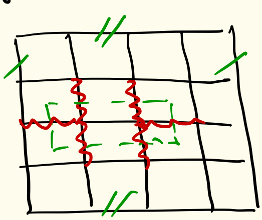
$|\psi_{10}\rangle =$ 

$|\psi_{01}\rangle =$  $+$  $+$  $+ \ldots$

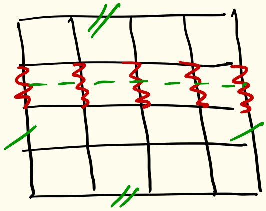
$|\psi_{11}\rangle =$  $+$  $+$  $+ \ldots$

Large vs small guage transformations:

If one were to instead work with a guage theory i.e. $H = H_p$ with the Gauss law constraint $\prod_{+} X = 1$ imposed on the Hilbert space, then the states related by guage transformation are identical. The ground states on torus are ofcourse identical to what we found above. The different ground states have the same (i.e. zero) flux through each plaquette (in the Z-basis), but cannot be related to each other by a local guage transformation. e.g. there is no way to go from a config. such as:



to



by a local ("small") gauge transformation.

Such transformations that relate these two configs. are called 'large guage transformations'.

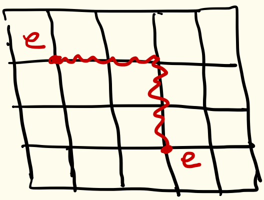
Excitations above the ground state:

There are two kinds of excitations in the toric code: electric charge (violation of H_s) and magnetic flux (violation of H_p).

The electric charge e sits on a vertex

and magnetic flux m " " " plaquette.

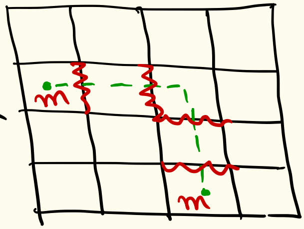
e in X basis:



Thus the e-excitations are created by an operator $\prod Z_i$ along a line which flips $X \to -X$ along the line and creates e-charges at the end-points of the line.

m in Z-basis :

Thus the m excitations are created by an operator $\prod X$ along a line on the dual-lattice, and it creates two m-excitations, located at the end plaquettes of the line.

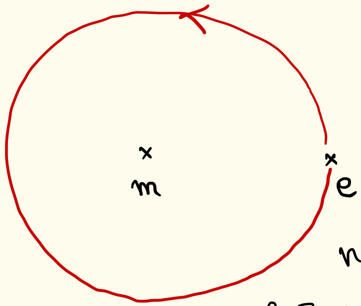


Self - Statistics of e and m excitations :

As discussed above, an e excitation located at location $\vec{r}$ can be moved to $\vec{r}'$ by an operator of the form $\prod_{l} Z_l$ where the line l goes from $\vec{r}$ to $\vec{r}'$. Since all Z_l operators commute with each other, exchanging two e excitations does not yield any phase / sign $\Rightarrow$ e excitations are self-bosons.

Similarly m excitations are moved around by operator of the form $\prod_{l} X$ and by same reasoning, they are also self-bosons.

Mutual statistics between e and m particles:



Consider an initial state $|e,m\rangle$ where the e, m particles are localized at certain positions. One now drags the e particle around the m particle and brings it back to its original location (m particle's position is unchanged during this process).

The new wave-fn is $\prod_{\mathcal{D}} Z_l |e,m\rangle$ where $\prod_{\mathcal{D}} Z$ acts along the path followed by the e-particle.

By Stokes thm. this is same as:

$$\prod_{\square \in \mathcal{D}} \left(\prod_{\square} Z_l \right) |e,m\rangle$$ where the product is taken over all plaquettes inside the loop.

Since there is a single m particle inside the loop, one and only one plaquette satisfies

$\prod_{\square} Z \, |e,m\rangle = -|e,m\rangle$ while for all others

$\prod_{\square} Z \, |e,m\rangle = +|e,m\rangle \Rightarrow$

$$\prod_{\square \in \mathcal{D}} \left(\prod_{\square} Z\right) |e, m\rangle = -|e, m\rangle.$$

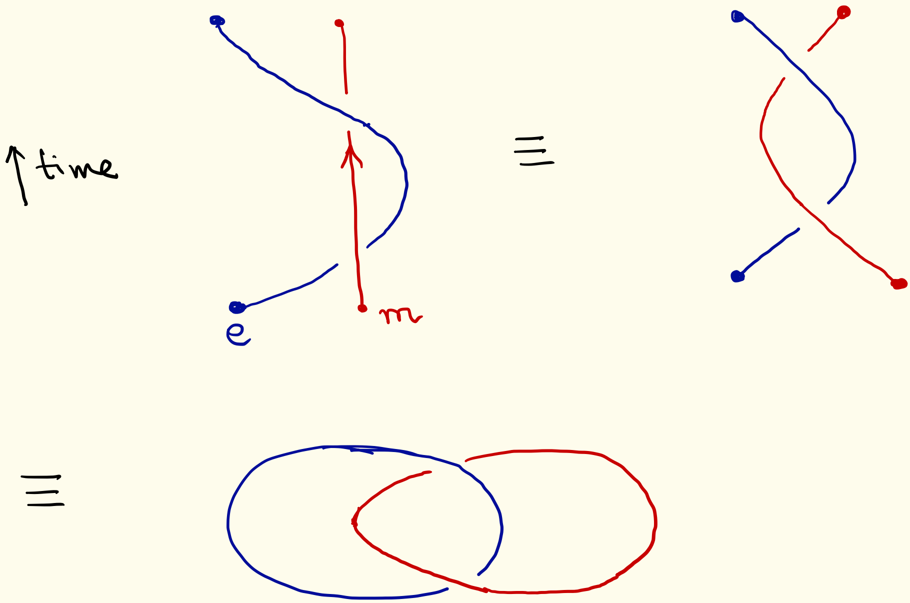
Therefore, the e and the m particles have a mutal statistics of -1. One needs to be a bit careful with the above argument. To be precise, one should compare the Berry phase for a process when e-particle goes around the same path as above, but <u>without</u> any m-particle inside the loop. The relative sign of these two processes is the Berry phase corresponding to braiding.

However, in this specific problem, if ~~there~~ was no m-particle inside the loop, then

$$\prod_{\square \in \mathcal{D}} \left(\prod_{\square} Z\right) |e, m\rangle = +|e, m\rangle$$ by the

same argument as above. Thus, the relative Berry phase - and hence the braiding statistics is indeed -1.

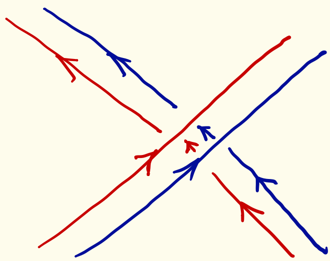
Space-time picture of braiding:



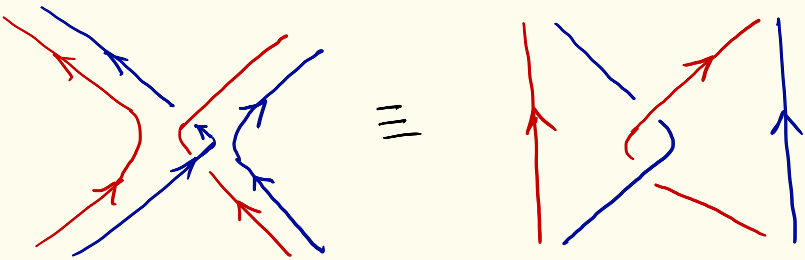
Thus, braiding corresponds to interlinked trajectories in space-time.

Self-statistics of the bound state of e+m :

To determine this, we need to exchange a bound state of em with another bound state of em :



Topologically this is same as (because both e and m are self-bosons)



$\Rightarrow$ the exchange of two em bound states yields the same sign as taking one e particle around another m particle ($\equiv$ braiding).

$\Rightarrow$ One picks up a sign of -1 under exchange

$\Rightarrow$ em bound state is a fermion!